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On the p-adic meromorphy of the function field height zeta function

C. Douglas Haessig

University of Rochester, New York, USA

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Abstract

In this brief note, we will investigate the number of points of bounded height in a projective variety defined over a function field, where the function field comes from a projective variety of dimension greater than or equal to 2. A first step in this investigation is to understand the p-adic analytic properties of the height zeta function. In particular, we will show that for a large class of projective varieties this function is p-adic meromorphic.

1. Introduction

Number fields. Let K be a number field of degree m and denote by H_K the relative multiplicative height function on the K-rational points in projective space $\mathbb{P}^n(K)$. Consider the number of K-rational points of height bounded above by d:

$$N_d(K) := \#\{x \in \mathbb{P}^n(K) \mid H_K(x) \leqslant d\}.$$

An asymptotic estimate for $N_d(K)$ was provided by Schanuel [2], who proved

$$N_d(K) = a(K,n)d^{n+1} + \begin{cases} \mathcal{O}(d \log d) & \text{if } K = \mathbb{Q} \text{ and } n = 1, \\ \mathcal{O}(d^{n+1-\frac{1}{m}}) & \text{otherwise,} \end{cases}$$

E-mail address: chaessig@math.rochester.edu.

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where

$$a(K,n) = \frac{hR/\omega}{\zeta_K(n+1)} \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|D_K|}} \right)^{n+1} (n+1)^{r_1+r_2-1},$$

h is the class number of K, R the regulator, ω the number of roots of unity in K, ζ_K the Dedekind zeta function of K, D_K the discriminant, and r_1, r_2 the number of real and complex embeddings of K.

Function fields. A function field analogue of Schanuel's result is also known. Let $\mathbb{F}_q$ be the finite field with q elements, $q = p^r$. Let X be a non-singular projective curve of genus g defined over $\mathbb{F}_q$, and let $\mathbb{F}_q(X)$ denote its function field. With an $\mathbb{F}_q(X)$-rational point $y := [y_0 : y_1 : \cdots : y_n] \in \mathbb{P}^n(\mathbb{F}_q(X))$ we may define the (logarithmic) height

$$h_X(y) := -\deg \inf_i (y_i)$$

where $\inf_i(y_i)$ denotes the greatest divisor D of X such that $D \leqslant (y_i)$ for all i with $y_i \neq 0$. As before, consider the number of $\mathbb{F}_q(X)$-rational points of height d:

$$N_d(\mathbb{P}^n) := \#\{ y \in \mathbb{P}^n(\mathbb{F}_q(X)) \mid h_X(y) = d \}.$$

An asymptotic estimate for $N_d(\mathbb{P}^n)$ was described by Serre [1, p. 19], and improved upon by Wan [4] as follows: for any $\epsilon > 0$,

$$N_d(\mathbb{P}^n) = \frac{hq^{(n+1)(1-g)}}{\zeta_X(n+1)(q-1)} q^{(n+1)d} + \mathcal{O}\big(q^{\frac{d}{2}+\epsilon}\big). \tag{1}$$

Wan's proof began by demonstrating that the associated function field height zeta function of $\mathbb{P}^n$ is a rational function. This meant, for all d, $N_d(\mathbb{P}^n) = \sum \beta_j^d - \sum \alpha_i^d$ where α_i and β_j are the zeros and poles of this rational function. The next step was to analyze these zeros and poles to arrive at (1).

Function field variations. Two natural variations of the above study are:

1. Consider X with $\dim(X) \geqslant 2$.
2. Replace $\mathbb{P}^n$ with an arbitrary projective variety Y defined over $\mathbb{F}_q(X)$.

Let us be more precise. Let X be a projective integral scheme of finite type over $\mathbb{F}_q$ and denote by $\mathbb{F}_q(X)$ its function field. Let $Y \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q(X)}$ be a projective variety defined over $\mathbb{F}_q(X)$. Define the height zeta function of $Y/\mathbb{F}_q(X)$ as

$$Z_{\mathrm{ht}}\big(Y(\mathbb{F}_q(X)), T\big) := \sum_{d=0}^{\infty} N_d(Y) T^d$$

where

$$N_d(Y) := \#\{ y \in Y(\mathbb{F}_q(X)) \mid h_X(y) = d \}.$$

The first step in the study of $N_d(Y)$ is to determine the (p-adic) analytic behavior of the height zeta function. When $Y = \mathbb{P}^n_{\mathbb{F}_q(X)}$ and the abelian group $A(X)$ of divisors of X modulo linear equivalence is of rank one, Wan [4] proved that $Z_{\mathrm{ht}}(\mathbb{P}^n(\mathbb{F}_q(X)), T)$ is p-adic meromorphic, and rational when $\dim(X) = 1$.

In this paper, under the same assumption that $A(X)$ has rank one, we will demonstrate that for a large class of projective varieties Y defined over $\mathbb{F}_q(X)$ with $\dim(X) \geqslant 2$, the height zeta function is p-adic meromorphic. As an example, suppose X is a complete intersection with $\dim(X) \geqslant 3$ and consider the projective hypersurface Y defined by $F := y_0^r G(y_1, \ldots, y_n)$ where G is a homogeneous polynomial in $\mathbb{F}_q(X)[y_1, \ldots, y_n]$ of degree d. As a consequence of Theorems 1 and 2, if $n > d^{\dim(X)+1}$ then $Z_{\mathrm{ht}}(Y(\mathbb{F}_q(X)), T)$ is p-adic meromorphic. Using Theorem 3, a similar result holds when Y is the locus of a system of homogeneous polynomials.

Our approach will be to reduce the study of the height zeta function to the study of two other zeta functions: the zeta functions of divisors and the Riemann–Roch zeta function. The zeta function of divisors has been studied in [5–7] and is expected to have a good p-adic theory. The Riemann–Roch zeta function is new, and Section 3 explores a few of its properties.

2. Reduction of the height zeta function

In this section, we will reduce the study of the function field height zeta function to the study of two other zeta functions, namely the zeta function of divisors and the Riemann–Roch zeta function.

Define $\mathrm{Div}(X/\mathbb{F}_q)$ as the free abelian group generated by the irreducible $\mathbb{F}_q$-subvarieties of X of codimension 1; we call these prime divisors. We say a divisor $D \in \mathrm{Div}(X/\mathbb{F}_q)$ is effective if its unique decomposition into prime divisors P_i, written $D = \sum n_i P_i$, has $n_i \geqslant 0$ for each i. Let $E(X)$ denote the monoid of effective divisors of X. Effectiveness places a partial order on the group $\mathrm{Div}(X/\mathbb{F}_q)$ as follows: we say $E \geqslant D$ if $E - D$ is effective. Further, for any set of divisors $\{D_1, \ldots, D_k\}$, we may define $\sup_i D_i$ as the smallest divisor D such that every $D - D_i$ is effective. Similarly, we may define $\inf_i D_i$ as the greatest divisor D such that every $D_i - D$ is effective.

The *zeta function of divisors* of X is defined by

$$Z_{\mathrm{div}}(X, T) := \sum_{D \in E(X)} T^{\deg(D)} = \prod_{P \text{ prime divisor}} \frac{1}{1 - T^{\deg(P)}}$$

where $\deg(D)$ is the projective degree of the divisor; see [7] for the definition. Note, when X is a curve, then $Z_{\mathrm{div}}(X, T)$ is the usual Weil zeta function, and hence is rational. When $\dim(X) \geqslant 2$, then $Z_{\mathrm{div}}(X, T)$ is never rational [5, Theorem 3.3].

Let $A(X)$ denote the abelian group of divisors of X modulo linear equivalence. Define $A^+(X) \subset A(X)$ as the monoid of effective divisor classes. Wan [5] has conjectured that when the rank of $A^+(X)$ is finite, then $Z_{\mathrm{div}}(X, T)$ is p-adic meromorphic. This was proven by Wan [5, Theorem 4.3] when the rank is one. Examples of p-adic meromorphy are also known for rank greater than one.

Next, let $W \hookrightarrow \mathbb{A}^n_{\mathbb{F}_q(X)}$ be an affine variety defined over $\mathbb{F}_q(X)$. For each divisor D of X, define the Riemann–Roch space $L(D)$ as the finite dimensional $\mathbb{F}_q$-vector space

$$L(D) := \bigl\{ f \in \mathbb{F}_q(X) \bigm| (f) + D \geqslant 0 \bigr\} \cup \{0\}.$$

Denote its dimension by $l(D)$. The *Riemann–Roch zeta function* of W over $\mathbb{F}_q(X)$ is defined by

$$Z_{\mathrm{RR}}\big(W\big(\mathbb{F}_q(X)\big), T\big) := \sum_{D \in E(X)} S_D T^{\deg(D)}$$

where S_D denotes the number of $\mathbb{F}_q(X)$-rational points of W whose affine coordinates lie in $L(D)$; that is,

$$S_D := \#\big(L(D)^n \cap W\big(\mathbb{F}_q(X)\big)\big) = \#\big\{f := (f_1, \dots, f_n) \in L(D)^n \bigm| f \in W\big(\mathbb{F}_q(X)\big)\big\}.$$

Our main theorem in this paper, proven in the next section, will show that the Riemann–Roch zeta function is often p-adic meromorphic.

Question 1. Assuming the rank of $A^+(X)$ is finite, is the Riemann–Roch zeta function always p-adic meromorphic?

Question 2. If X is a curve, is the Riemann–Roch zeta function always rational?

We are now able to reduce the height zeta function as a sum of the two zeta functions above. First, recall the following decomposition of projective space. Denote by $y = [y_0 : y_1 : \cdots : y_n]$ the points of $\mathbb{P}^n_{\mathbb{F}_q(X)}$. For each i, define the affine space $\mathbb{A}^{n-i}_{\mathbb{F}_q(X)} \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q(X)}$ as the variety defined by setting $y_0 = \cdots = y_{i-1} = 0$ and $y_i = 1$. We then have the decomposition of projective space into a disjoint union:

$$\mathbb{P}^n_{\mathbb{F}_q(X)} = \mathbb{A}^n_{\mathbb{F}_q(X)} \cup \mathbb{A}^{n-1}_{\mathbb{F}_q(X)} \cup \cdots \cup \mathbb{A}^0_{\mathbb{F}_q(X)}.$$

Let $Y \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q(X)}$ be a projective variety. For each i we may define the affine algebraic variety $Y_i := Y \cap \mathbb{A}^i_{\mathbb{F}_q(X)}$.

Theorem 1. *We may decompose the height zeta function as follows*:

$$Z_{\mathrm{ht}}\big(Y\big(\mathbb{F}_q(X)\big), T\big) = \frac{1}{Z_{\mathrm{div}}(X, T)} \sum_{i=0}^{n} Z_{\mathrm{RR}}\big(Y_i\big(\mathbb{F}_q(X)\big), T\big).$$

Proof. Since every coordinate of $y = [y_0 : y_1 : \cdots : y_n] \in \mathbb{P}^n_{\mathbb{F}_q(X)}$ cannot be simultaneously zero, we may assume $y_i = 1$ for some i. It follows that $-\inf_i(y_i) = \sup_i(y_i)_\infty$, where $(y_i)_\infty$ is the polar divisor of $y_i \in \mathbb{F}_q(X)$. Thus,

$$\begin{aligned}
Z_{\mathrm{ht}}\big(Y\big(\mathbb{F}_q(X)\big), T\big) &= \sum_{[y_0:\cdots:y_n] \in Y} T^{-\deg \inf_i(y_i)} \\
&= \sum_{i=0}^{n} \sum_{(y_0,\dots,y_n) \in Y_i} T^{\deg \sup(y_i)_\infty} \\
&= \sum_{i=0}^{n} \sum_{D \in E(X)} H_D^{(i)} T^{\deg(D)}
\end{aligned}$$

where $H_D^{(i)} := \#\{(y_0, \ldots, y_n) \in Y_i \mid \sup(y_i)_\infty = D\}$; note, we put round-brackets to indicate that the point $(y_0, \ldots, y_n)$ lies in affine space. Now,

$$Z_{\mathrm{div}}(X,T)\left(\sum_{D\in E(X)} H_D^{(i)} T^{\deg(D)}\right) = \sum_{E\in E(X)}\left(\sum_{0\leqslant D\leqslant E} H_D^{(i)}\right) T^{\deg(E)}.$$

The theorem follows since $S_E(Y_i) = \sum_{0\leqslant D\leqslant E} H_D^{(i)}$. □

3. The Riemann–Roch zeta function

In this section, we will study the p-adic meromorphy of the Riemann–Roch zeta function. Our main result will be to show, for a large class of affine varieties, that it is p-adic entire. After this, we present some explicit examples.

Theorem 2. *Suppose the abelian group $A(X)$ has rank one, and* $\dim(X) \geqslant 2$. *Let $Y \hookrightarrow \mathbb{A}^n_{\mathbb{F}_q(X)}$ be an affine hypersurface defined by a polynomial $F \in \mathbb{F}_q(X)[y_1, \ldots, y_n]$. Define $d := \deg(F)$. If $n > d^{\dim(X)+1}$, then $Z_{\mathrm{RR}}(Y(\mathbb{F}_q(X)), T)$ is a p-adic entire function.*

Proof. We begin by writing

$$Z_{\mathrm{RR}}\big(Y\big(\mathbb{F}_q(X)\big), T\big) := \sum_{D\in E(X)} S_D T^{\deg(D)} = \sum_{k\geqslant 0} M_k T^k$$

where

$$M_k := \sum_{D\in E(X),\, \deg(D)=k} S_D. \tag{2}$$

We will show that under the given hypothesis, for all k sufficiently large, $\mathrm{ord}_q(S_D)$ is bounded below by a polynomial of degree $\dim(X)$. Hence, the same is true for $\mathrm{ord}_q(M_k)$ proving the theorem.

Decomposing $A(X)$ into its torsion part and free part, we have $A(X) = \{D_1^*, \ldots, D_h^*\}_{\mathrm{tor}} \oplus \mathbb{Z}D$ for some divisor $D > 0$. Let $\mu := \deg(D)$. Thus, every effective divisor E of X has degree $k\mu$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Further, if $\deg(E) = k\mu$, then E is linearly equivalent to $D_j^* + kD$ for some j. By [7, Proposition 2], for $k \gg 0$ and $\deg(E) = k\mu$, the dimension of the Riemann–Roch space satisfies:

$$l(E) = l\big(D_j^* + kD\big) = ck^{\dim(X)} + O\big(k^{\dim(X)-1}\big)$$

where $c := \frac{D^{\dim(X)}}{\dim(X)!}$. From now on, we will assume k is large enough so that the above formula holds for any effective divisor of degree $k\mu$.

Next, let us write $F = \sum_{j=1}^{J} \alpha_j y^{v_j}$ where $\alpha_j \in \mathbb{F}_q(X)$ and $v_j \in \mathbb{Z}^n_{\geqslant 0}$. With the effective divisor $W := \sup_j(\alpha_j)_\infty$, we have $\alpha_j \in L(W)$ for every $j = 1, \ldots, J$. Find i and ρ such that $D_i^* + \rho D$ is linearly equivalent to W. Using this, we observe that if $(f_1, \ldots, f_n) \in L(E)^n$, then $F(f_1, \ldots, f_n) \in L(dE + W)$.

Let $u_1, \ldots, u_s$ be a basis of $L(E)$ over $\mathbb{F}_q$. Thus, for every $h \in L(E)$, there exists unique $x_i \in \mathbb{F}_q$ such that $h = x_1 u_1 + \cdots + x_s u_s$, and vice versa. Using this structure, we may rewrite the polynomial $F(y_1, \ldots, y_n)$ with the substitution

$$\begin{aligned} y_1 &= x_1^{(1)} u_1 + \cdots + x_s^{(1)} u_s, \\ y_2 &= x_1^{(2)} u_1 + \cdots + x_s^{(2)} u_s, \\ &\vdots \\ y_n &= x_1^{(n)} u_1 + \cdots + x_s^{(n)} u_s. \end{aligned}$$

Thus, we now have F as a function in the variables $x_i^{(j)}$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant n$. If we let $w_1, \ldots, w_r$ be a basis of $L(dE + W)$, then we may write

$$F\big(x_i^{(j)}\big) = g_1\big(x_i^{(j)}\big) w_1 + \cdots + g_r\big(x_i^{(j)}\big) w_r$$

for some $g_i \in \mathbb{F}_q[x_i^{(j)}\colon 1 \leqslant i \leqslant s \text{ and } 1 \leqslant j \leqslant n]$. Notice that $\deg(g_i) \leqslant d$. Writing F in this form means that elements of S_E are precisely the elements in the zero locus of the set $\{g_1, \ldots, g_r\}$ in the affine space $\mathbb{A}^{sn}_{\mathbb{F}_q}$. Therefore, by a theorem of Ax–Katz [3]

$$\mathrm{ord}_q(S_E) \geqslant \frac{sn - \sum_{i=1}^r \deg(g_i)}{\max \deg(g_i)}.$$

Since, $s = l(E)$ and $r = l(dE + W)$, and recalling that E and W are linearly equivalent to $D_j^* + kD$ and $D_i^* + \rho D$ respectively, we see that

$$\begin{aligned} \mathrm{ord}_q(S_E) &\geqslant \frac{sn - \sum_{i=1}^r \deg(g_i)}{\max \deg(g_i)} \\ &\geqslant \frac{1}{d}\big[l(E)n - l(dE + W)d\big] \\ &= \frac{1}{d}\big[l\big(D_j^* + kD\big)n - l\big(\big(dD_j^* + D_i^*\big) + (dk + \rho)D\big)d\big] \\ &= \frac{1}{d}\big[ck^{\dim(X)}n - c(kd + \rho)^{\dim(X)}d\big] + O\big(k^{\dim(X)-1}\big) \\ &= \frac{ck^{\dim(X)}}{d}\big[n - d^{\dim(X)+1}\big] + O\big(k^{\dim(X)-1}\big). \end{aligned}$$

The theorem follows. □

Remark. When X is a curve and $n > d^{\dim(X)+1}$, the proof above demonstrates that the Riemann–Roch zeta function is p-adic entire in a disk of radius $1 + \epsilon$. A similar result is expected when X is a projective toric variety and $n > d^{\mathrm{rank}(A(X)) \dim(X)+1}$.

Using the system of equations version of Ax–Katz [3] and a similar proof to the one above, we may extend the results of Theorem 2 as follows:

Theorem 3. *Suppose the abelian group $A(X)$ has rank one, and $\dim(X) \geqslant 2$. Let $Y \subset \mathbb{A}^n_{\mathbb{F}_q(X)}$ be the affine variety defined by the polynomials $F_1, \ldots, F_r \in \mathbb{F}_q(X)[y_1, \ldots, y_n]$. Define $d_i := \deg F_i$. Suppose $n > \sum_{i=1}^r d_i^{\dim(X)+1}$. Then $Z_{\mathrm{RR}}(Y(\mathbb{F}_q(X)), T)$ is a p-adic entire function.*

Example. Consider affine n-space $\mathbb{A}^n_{\mathbb{F}_q(X)}$. In this case, we have

$$Z_{\mathrm{RR}}\big(\mathbb{A}^n\big(\mathbb{F}_q(X)\big), T\big) = \sum_{D \geqslant 0} q^{nl(D)} T^{\deg(D)}. \tag{3}$$

It follows that if $A(X)$ has rank one, then this is a p-adic entire function when $\dim(X) \geqslant 2$ and a rational function when X is a curve.

Wan [4] has conjectured that when the rank of $A^+(X)$ is finite, then the height zeta function of projective space $\mathbb{P}^n_{\mathbb{F}_q(X)}$ is p-adic meromorphic. An equivalent conjecture is that the Riemann–Roch zeta function of affine space $\mathbb{A}^n_{\mathbb{F}_q(X)}$ is p-adic meromorphic.

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